

Asymptotic behavior for the heat equation in nonhomogeneous media with critical density

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Abstract

We study the large-time behavior of solutions to the heat equation in nonhomogeneous media with critical singular density

$$|x|^{-2}\partial_t u = \Delta u, \quad \text{in } \mathbb{R}^N \times (0, \infty)$$

in dimensions $N \geq 3$. The asymptotic behavior proves to have some interesting and quite striking properties. We show that there are two completely different asymptotic profiles depending on whether the initial data u_0 vanishes at $x = 0$ or not. Moreover, in the former the results are true only for radially symmetric solutions, and we provide counterexamples to convergence to symmetric profiles in the general case.

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1 Introduction

The aim of this work is to establish the asymptotic behavior of solutions to the following heat equation in nonhomogeneous media with critical density:

$$|x|^{-2}\partial_t u(x, t) = \Delta u(x, t), \quad (x, t) \in \mathbb{R}^N \times (0, \infty). \quad (1.1)$$

as a part of an ongoing project of studying the asymptotic behavior for

$$|x|^{-2}\partial_t u(x, t) = \Delta u^m(x, t), \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \quad (1.2)$$

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with $m \geq 1$. The technically more involved problem with the large-time behavior for $m > 1$ will be studied in a forthcoming paper [8].

Equations of type (1.2) with general densities have been proposed by Kamin and Rosenau in a series of papers [11, 12, 13] to model thermal propagation by radiation in non-homogeneous plasma. Since then, many papers were devoted to developing rigorously the qualitative theory or asymptotic behavior for

$$\varrho(x)\partial_t u(x, t) = \Delta u^m(x, t), \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \quad (1.3)$$

usually asking that $\varrho(x) \sim |x|^{-\gamma}$ as $|x| \rightarrow \infty$, for some $\gamma > 0$ (e.g. [18]). Thus, for $m > 1$, it has been noticed that while $0 < \gamma < 2$, the solutions have similar properties to the ones of the pure porous medium equation (for short, PME)

$$\partial_t u = \Delta u^m,$$

see [17, 19], while for $\gamma > 2$, the properties of the solutions depart strongly from the ones of the PME [10]. Thus, $\gamma = 2$ is critical, and the asymptotic behavior for this case is left open in [10] with a conjecture giving the explicit profile. On the other hand, the asymptotic behavior for (1.3) with a density $\varrho(x) \sim |x|^{-2}$ at infinity, but ϱ regular near the origin, is studied in another recent work [15], obtaining an explicit profile solving (1.2), and proving the convergence towards it in the outer region (that is, outside small compacts near $x = 0$). Unusually, the linear diffusion problem $m = 1$ has been studied later than its nonlinear version, see for example [2, 3].

In all cases, it has been noticed that the solutions converge asymptotically towards profiles coming from the pure power density equation, that is, $\varrho(x) = |x|^{-\gamma}$. Moreover,

$$|x|^{-\gamma}\partial_t u(x, t) = \Delta u^m(x, t) \quad (1.4)$$

has some more interesting feature: a singularity at $x = 0$, apart from the decay at infinity. Recently, in [7] the authors study formally some properties of radial solutions to (1.4) as a first step to understand its general behavior. Moreover, a study of existence and uniqueness for (1.4) for $\gamma > 2$ and $m > 1$ is done in [10, Section 6].

Coming back to our problem, that of letting $m = 1$, $\gamma = 2$ in (1.4), we find explicit asymptotic profiles which explain better the effect of the singularity at $x = 0$. The general case $m > 1$ will be treated in the companion paper [8]. But in order to explain these comments and to make precise the motivation for this work, let us state the main results of the paper.

Main results. We deal with the Cauchy problem associated to Eq. (1.1) with initial data

$$u_0 \in L^1_2(\mathbb{R}^N), \quad u_0 \geq 0, \quad (1.5)$$

where the dimension is $N \geq 3$ (except when specified) and

$$L^1_2(\mathbb{R}^N) := \left\{ h : \mathbb{R}^N \mapsto \mathbb{R}, \text{ } h \text{ measurable, } \int_{\mathbb{R}^N} |x|^{-2} h(x) dx < \infty \right\}.$$

In Section 2 we give the precise notions of *weak solution* and *strong solution* to (1.1) and we prove that the Cauchy problem for (1.1) is well-posed in our framework. We refer the interested reader to Definition 2.1 and Theorem 2.2 for the precise statements.

We state the results about the large-time behavior, that we find quite interesting and unexpected. We begin with the case when $u_0(0) = 0$.

Theorem 1.1. *Let u be a **radially symmetric** solution of Eq. (1.1) with initial data satisfying (1.5) and moreover*

$$M_{u_0} := \int_{\mathbb{R}^N} |x|^{-N} u_0(x) dx < \infty, \quad u_0(0) = 0. \quad (1.6)$$

Then we have

$$\lim_{t \rightarrow \infty} t^{1/2} \left\| u(x, t) - \frac{M_{u_0}}{\omega_1} F(x, t) \right\|_{\infty} = 0, \quad (1.7)$$

where ω_1 is the area of the unit sphere in \mathbb{R}^N and

$$F(x, t) := \begin{cases} \frac{1}{\sqrt{4\pi t}} G\left(\frac{\log|x| + (N-2)t}{2\sqrt{t}}\right), & G(\xi) = e^{-\xi^2}, \quad \text{for } |x| \neq 0, \\ 0, & \text{for } x = 0, \end{cases} \quad (1.8)$$

Remarks. (a) Let us notice also that

$$\max\{F(x, t) : x \neq 0\} = O(t^{-1/2}), \quad \text{as } t \rightarrow \infty,$$

showing that the time-scale $t^{1/2}$ is the correct one for the asymptotic behavior in (1.7). More precisely,

$$\|F(\cdot, t)\|_{\infty} = \frac{1}{\sqrt{4\pi t}}, \quad (1.9)$$

as it will be analyzed in the remarks at the end of Section 4.

(b) The mass M_{u_0} in (1.6) is conserved along the flow. This will be obvious from the proof of Theorem 1.1.

Counterexamples in the non-radial case. We emphasize on the fact that this is true *only for radially symmetric solutions*. For general solutions, Theorem 1.1 is not true. It is quite surprising that, due to the singularity at $x = 0$, the solutions that start from $u_0(0) = 0$ do not converge asymptotically to a radial profile.

We construct a counterexample to Theorem 1.1 for solutions that are not radially symmetric. We pass to generalized spherical coordinates $x = (r, \phi_1, \dots, \phi_{N-2}, \theta)$ in \mathbb{R}^N and define the following function

$$F_N(x, t) = F_N(r, \phi_1, \dots, \phi_{N-2}, \theta) := \theta F(r, t).$$

Using the formula of the Laplace operator in spherical coordinates, from the particular form of F_N (that involves no dependence on $(\phi_1, \dots, \phi_{N-2})$ and $\partial_{\theta}^2 F_N \equiv 0$), we find that F_N is a solution to (1.1), since

$$\Delta F_N(x, t) = \theta \Delta_r F(r, t) = \theta |x|^{-2} F_t(r, t) = |x|^{-2} \partial_t F_N(x, t),$$

where Δ_r is the Laplacian operator in radial variable. The fact that Theorem 1.1 is not satisfied on this example is obvious since (1.7) becomes

$$\lim_{t \rightarrow \infty} t^{1/2} \|(\theta - K)F(r, t)\|_{\infty} = 0,$$

for some constant K depending on the dimension N , which is false since θ is variable, K is constant and $t^{1/2} \|F(\cdot, t)\|_{\infty} = 1/\sqrt{4\pi}$.

When the value of the initial data at the origin is nonzero, things are completely different, as the following theorem states.

Theorem 1.2. *Let u be a **general** solution of Eq. (1.1) with initial data u_0 satisfying (1.5) and $u_0(0) = K > 0$. If there exist $\delta, \varepsilon > 0$ such that the initial data u_0 satisfies*

$$|K - u_0(x)| \leq |x|^{\delta}, \quad \text{as } |x| \rightarrow 0, \quad u_0(x) \leq |x|^{-\varepsilon}, \quad \text{as } |x| \rightarrow \infty, \quad (1.10)$$

then we have

$$\left\| u(t) - \frac{K}{2} E(t) \right\|_{\infty} = O(t^{-1/2}), \quad \text{as } t \rightarrow \infty, \quad (1.11)$$

where

$$E(x, t) := \begin{cases} \operatorname{erfc}\left(\frac{\log|x| + (N-2)t}{2\sqrt{t}}\right), & \operatorname{erfc}(\xi) = \frac{2}{\sqrt{\pi}} \int_{\xi}^{\infty} e^{-\theta^2} d\theta, \quad \text{for } |x| \neq 0, \\ K, & \text{for } x = 0. \end{cases} \quad (1.12)$$

Remarks. 1. Theorem 1.2 can be improved by relaxing and generalizing the condition (1.10); indeed, (1.11) holds true if there exists some functions $\Phi_1, \Phi_2 : (0, \infty) \mapsto (0, \infty)$ such that

$$\int_{0+} \frac{\Phi_1(r)}{r} dr < \infty, \quad \int^{+\infty} \frac{1}{r\Phi_2(r)} dr < \infty,$$

and

$$u_0(x) \leq \Phi_1(|x|), \quad \text{as } x \rightarrow 0, \quad u_0(x) \leq \Phi_2(|x|), \quad \text{as } |x| \rightarrow \infty.$$

The proof is totally similar.

2. Theorem 1.2 is indeed true in general, also without radial symmetry, in contrast with Theorem 1.1. These two different results show the importance of the singularity of the density $|x|^{-2}$. Indeed, the evolution gives rise in a striking way to two different asymptotic profiles, only depending on a difference in the pointwise value of u_0 at the origin.

When dealing with radially symmetric solutions, the results can be improved both with respect to the condition on the initial data u_0 and with respect to the rate of convergence. This is gathered in the following

Theorem 1.3. *Let u be a **radially symmetric** solution for Eq. (1.1) whose initial data satisfies (1.5).*

(a) *The asymptotic convergence (1.11) holds true under the following condition on u_0 :*

$$I_1 := \int_{B(0,1)} |x|^{-N} |K - u_0(x)| dx + \int_{\mathbb{R}^N \setminus B(0,1)} |x|^{-N} |u_0(x)| dx < \infty. \quad (1.13)$$

(b) *If we ask furthermore the following condition on the gradient*

$$I_2 := \int_{\mathbb{R}^N} \frac{|(\log |x|)^3|}{|x|^{N-1}} |\nabla u_0(x)| dx < \infty, \quad (1.14)$$

then we have a rate of convergence:

$$\left\| u(t) - \frac{K}{2} E(t) \right\|_{\infty} = O(t^{-3/2}), \quad \text{as } t \rightarrow \infty. \quad (1.15)$$

We will add here two more interesting consequences about the behavior near $x = 0$, where the singularity at $x = 0$ plays a key role.

Proposition 1.4. (a) *There is no fundamental solution of Eq. (1.1), that is, a solution with initial data $u_0 = M\delta_0$.*

(b) *If the initial condition u_0 satisfies $u_0(x) = 0$ for any $x \in B(0, r_0)$, for some $r_0 > 0$, then $u(x, t) > 0$ for all $(x, t) \in (\mathbb{R}^N \setminus \{0\}) \times (0, \infty)$ and $u(0, t) = 0$ for any $t > 0$.*

We plot the asymptotic profiles in Figure 1 in dimension $N = 3$, the general picture being similar. In the pictures we see the profiles evolving at various times.

Organisation of the paper. Before passing to the asymptotic behavior, we begin by studying the well-posedness for (1.1) with suitable initial data. This is the goal of Section 2, which insures us that the object of our study exists in suitable spaces. We then prove the main results in two steps. In a first step, we describe a transformation mapping radially symmetric solutions of (1.1) into solutions of the one-dimensional heat equation. This is done in Section 3. With the aid of it, we prove Theorem 1.1 and part (a) of Theorem 1.3. Then, in order to prove part (b) of Theorem 1.3, we need one more transformation step. The proofs of the theorems for radially symmetric solutions and some more remarks about the asymptotic profiles are the subject of Section 4. Finally, in Section 5 we prove Theorem 1.2, using as essential tool a comparison principle proved in Section 2, and we end with the proof of Proposition 1.4. We close the paper with a section of comments where we include a brief discussion of the dimension $N = 1$ for (1.1) and raise some open problems.

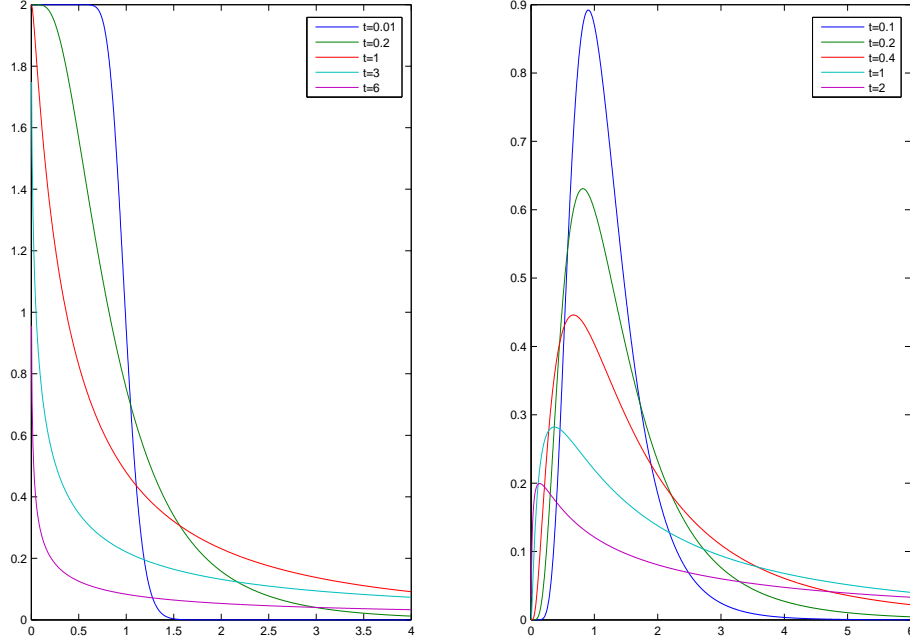


Figure 1: Profiles E and F for dimension $N = 3$ at various times.

2 Existence and uniqueness

Before proving the main results about asymptotic behavior, we have to develop a theory of existence and uniqueness for the Cauchy problem

$$\begin{cases} |x|^{-2}u_t = \Delta u, & \text{in } \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (2.1)$$

for suitable initial data $u_0 \in L^1_2(\mathbb{R}^N)$. We start from the precise definition of a solution.

Definition 2.1. *We say that a function $u(x, t)$ is a weak solution to (2.1) if it satisfies the following conditions:*

- (a) $u \in C([0, \infty); L^1_2(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N \times (\tau, \infty))$, for any $\tau > 0$, and $u \geq 0$ in $\mathbb{R}^N \times (0, \infty)$;
- (b) The integral identity

$$\int \int u(\Delta \varphi + |x|^{-2}\varphi_t) dx dt = 0 \quad (2.2)$$

holds true for any test function $\varphi \in C_0^\infty(\mathbb{R}^N \times (0, \infty))$, and $u(0) = u_0$.

We have the following:

Theorem 2.2. *Let $u_0 \in L^1_2(\mathbb{R}^N)$, u_0 nonnegative. Then there exists a unique weak solution u to the Cauchy problem (2.1) such that*

$$|x|^{-2}u_t, \Delta u \in L^1_{\text{loc}}(Q_*), \quad |x|^{-2}u_t = \Delta u \text{ a.e. in } Q_*,$$

where $Q_* = \mathbb{R}^N \times (0, \infty) \setminus \{(0, t) : t > 0\}$.

This type of solution is usually referred as a *strong solution* (see [20]). The proof will adapt ideas from the proof of existence in [10, Section 6], where well-posedness is proved for (1.4) with $\gamma > 2$ and $m > 1$, thus at some points we will be rather sketchy.

Proof of Theorem 2.2. Uniqueness. This is based on the following result which is also interesting by itself.

Proposition 2.3 (L^1_2 -Contraction principle). *Let u_1, u_2 be two strong solutions of Eq. (1.1). For $0 < t_1 < t_2$ we have*

$$\int_{\mathbb{R}^N} |x|^{-2} [u_1(x, t_2) - u_2(x, t_2)]_+ dx \leq \int_{\mathbb{R}^N} |x|^{-2} [u_1(x, t_1) - u_2(x, t_1)]_+ dx, \quad (2.3)$$

where $[g]_+$ represents the positive part.

Proof. We follow the same ideas as in [20, Proposition 9.1]. Let $p \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be such that $p(s) = 0$ for $s \leq 0$, $p'(s) > 0$ for $s > 0$ and $0 \leq p \leq 1$. Let j be the primitive of p such that $j(0) = 0$. The idea is to choose p as an approximation of the function

$$\text{sign}_0^+(r) = \begin{cases} 1, & \text{if } r > 0, \\ 0, & \text{if } r \leq 0, \end{cases}$$

thus j will approximate the positive part function. Consider also a cutoff function ξ_n constructed in the following way: let $\xi_0 \in C_0^\infty(\mathbb{R}^N)$, $0 \leq \xi_0 \leq 1$, $\xi_0(x) = 0$ for $|x| \geq 2$, $\xi_0(x) = 1$ for $|x| \leq 1$, and define $\xi_n(x) := \xi_0(x/n)$. Subtracting the equations satisfied by u_2 and u_1 , then multiplying by the test function $p(u_1 - u_2)\xi_n$, we obtain

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^N} |x|^{-2} (u_1 - u_2)_t p(u_1 - u_2) \xi_n dx dt = \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \Delta(u_1 - u_2) p(u_1 - u_2) \xi_n dx dt. \quad (2.4)$$

By approximation of $u_1 - u_2$ as in the proof of [20, Proposition 9.1] and integration by parts in the right-hand side of (2.4), we obtain, with $\xi = \xi_n$:

$$\begin{aligned} \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |x|^{-2} (u_1 - u_2)_t p(u_1 - u_2) \xi dx dt &\leq - \int_{t_1}^{t_2} \int_{\mathbb{R}^N} p(u_1 - u_2) \nabla(u_1 - u_2) \cdot \nabla \xi dx dt \\ &= - \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \nabla j(u_1 - u_2) \cdot \nabla \xi dx dt \\ &= \int_{t_1}^{t_2} \int_{\mathbb{R}^N} j(u_1 - u_2) \Delta \xi dx dt \leq \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |u_1 - u_2| |\Delta \xi| dx dt. \end{aligned}$$

We let now $p \rightarrow \text{sign}_0^+$ and integrate to get

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^{-2} [u_1(x, t_2) - u_2(x, t_2)]_+ \xi_n dx &\leq \int_{\mathbb{R}^N} |x|^{-2} [u_1(x, t_1) - u_2(x, t_1)]_+ \xi_n dx \\ &\quad + \|\Delta \xi_n\|_\infty \int_{t_1}^{t_2} \int_{|x| > n} |u_1 - u_2| dx dt. \end{aligned}$$

Letting now $n \rightarrow \infty$ and taking into account that we deal with solutions in L^1 , we obtain the result. \square

The uniqueness follows obviously from the contraction principle. Moreover, we have the following:

Corollary 2.4 (Comparison principle). *If u_1, u_2 are solutions of Eq. (1.1) such that their initial data satisfy $u_{0,1} \leq u_{0,2}$, then $u_1 \leq u_2$ in $\mathbb{R}^N \times (0, \infty)$.*

Proof of Corollary 2.4. Suppose that $u_{0,1}, u_{0,2} \in L^1_2(\mathbb{R}^N)$ satisfy $u_{0,1} \leq u_{0,2}$ and u_1, u_2 are the corresponding solutions. Then, for any $t > 0$, we have

$$\int_{\mathbb{R}^N} |x|^{-2} [u_1(x, t) - u_2(x, t)]_+ dx \leq \int_{\mathbb{R}^N} |x|^{-2} [u_{0,1}(x) - u_{0,2}(x)]_+ dx = 0,$$

hence $u_1(x, t) \leq u_2(x, t)$. \square

Existence. This part of the proof is more involved and will be divided, for a better comprehension, into several steps.

Step 1. Compactly supported data. In a first step, we prove existence of a strong solution for compactly supported initial data outside the origin. For $0 < r < R < \infty$, we denote an annulus by $B_{r,R} := B_R \setminus B_r$. Given $u_0 \in C^\infty_0(\mathbb{R}^N \setminus \{0\})$, consider the approximating Dirichlet problem

$$\begin{cases} |x|^{-2} u_t = \Delta u, & \text{in } Q_{r,R} := B_{r,R} \times (0, \infty), \\ u(x, 0) = u_0(x), & \text{in } B_{r,R}, \\ u(x, t) = 0, & \text{in } \partial B_{r,R} \times (0, \infty). \end{cases} \quad (2.5)$$

The problem (2.5), being not singular at $x = 0$ neither degenerate at infinity, admits a unique solution $u_{r,R}$ as shown in [4, 5]. Moreover, by simple comparison in the smaller annulus, it is easily seen that $u_{r_2,R_2} \geq u_{r_1,R_1}$ if $r_2 \leq r_1$ and $R_2 \geq R_1$. It remains to show that they are uniformly bounded. To this end, recall that $u_0 \in C^\infty_0(\mathbb{R}^N \setminus \{0\})$ and consider $r > 0$ small enough and $R > 0$ large enough such that $\text{supp } u_0 \subset B_{r,R}$. Recalling the definition of $F(x, t)$ given in (1.8), for some $\tau > 0$ sufficiently small and $K > 0$ large, we have

$$u_0(x) \leq KF(x, \tau) \quad \text{in } B_{r,R}.$$

Since on the lateral boundary $u_{r,R}(x, t) = 0 \leq KF(x, t + \tau)$ for any x such that either $|x| = r$ or $|x| = R$, by standard comparison we obtain the following important universal bound

$$u_{r,R}(x, t) \leq KF(x, t + \tau), \quad \text{in } B_{r,R} \times (0, \infty). \quad (2.6)$$

This is a uniform bound which does not depend on r, R . Thus, the following limit

$$u(x, t) = \lim_{r \rightarrow 0} \lim_{R \rightarrow \infty} u_{r,R}(x, t), \quad (2.7)$$

is well-defined and gives rise to a weak solution to (1.1), as it is easy to check with the definition. In this way, we have solved the problem for $u_0 \in C^\infty_0(\mathbb{R}^N \setminus \{0\})$. Moreover, let us notice here that any solution obtained through this approximation process also satisfies the universal bound (2.6) in Q_* .

Step 2. Preliminary estimates. We still want to prove that our weak solutions are in fact strong, then pass to the general case by some approximation process. To this end, we establish two estimates that will imply further regularity on the solutions. They are gathered in the following

Lemma 2.5. *Let u be a weak solution to (1.1) such that $u_0 \in C^\infty_0(\mathbb{R}^N \setminus \{0\})$. Let $K \subset \mathbb{R}^N \setminus \{0\}$ be a compact set and $0 < \tau < T < \infty$. Then there exists a positive constant $C > 0$ depending only on N , $\text{dist}(\{0\}, K)$ and τ such that*

$$\int_\tau^T \int_K |\nabla u|^2 dx dt \leq C \quad (2.8)$$

and

$$\int_\tau^T \int_K |x|^{-2} |u_t|^2 dx dt \leq C. \quad (2.9)$$

Proof. The proof adapts the ones in [10, Theorem 3.6, Theorem 3.8], but we give a complete version of it for the sake of completeness and simplicity. Let \overline{K} be a compact neighborhood of K such that 0 does not belong to \overline{K} and consider a cutoff function $\eta \in C_0^\infty(\mathbb{R}^N)$ such that

$$\eta \equiv 1 \text{ in } K, \quad \eta \equiv 0 \text{ in } \mathbb{R}^N \setminus \overline{K}, \quad 0 \leq \eta \leq 1 \text{ in } \mathbb{R}^N.$$

We multiply (1.1) by $u\eta^2$ and integrate by parts to get

$$\begin{aligned} \frac{d}{dt} \int_{\overline{K}} |x|^{-2} \frac{u^2}{2} \eta^2 dx &= \int_{\overline{K}} u \Delta u \eta^2 dx = - \int_{\overline{K}} \nabla u \cdot \nabla (u \eta^2) dx \\ &= - \int_{\overline{K}} |\nabla u|^2 \eta^2 dx - 2 \int_{\overline{K}} u \eta \nabla u \cdot \nabla \eta dx. \end{aligned}$$

Integrating on $[\tau, T]$ and using Young's inequality, we further obtain

$$\begin{aligned} \frac{1}{2} \int_{\overline{K}} |x|^{-2} u^2(x, T) \eta^2 dx + \int_{\tau}^T \int_{\overline{K}} |\nabla u|^2 \eta^2 dx dt &= \frac{1}{2} \int_{\overline{K}} |x|^{-2} u^2(x, \tau) \eta^2 dx - 2 \int_{\tau}^T \int_{\overline{K}} u \eta \nabla u \cdot \nabla \eta dx dt \\ &\leq \frac{1}{2} \int_{\overline{K}} |x|^{-2} u^2(x, \tau) \eta^2 dx + \frac{1}{2} \int_{\tau}^T \int_{\overline{K}} |\nabla u|^2 \eta^2 dx dt + 2 \int_{\tau}^T \int_{\overline{K}} u^2 |\nabla \eta|^2 dx dt, \end{aligned}$$

whence estimate (2.8) follows from the uniform bound (2.6), which implies readily an uniform bound for the terms in the right-hand side. Consequently,

$$\int_{\tau}^T \int_K |\nabla u|^2 dx dt \leq \int_{\tau}^T \int_{\overline{K}} |\nabla u|^2 \eta^2 dx dt \leq C$$

as desired.

In order to prove the second estimate, we multiply (1.1) this time by $u_t \eta$, then integrate in space and time. We obtain

$$\int_{\overline{K}} |x|^{-2} |u_t|^2 \eta dx = - \int_{\overline{K}} \nabla u \cdot \nabla (u_t \eta) dx = - \frac{1}{2} \frac{d}{dt} \int_{\overline{K}} |\nabla u|^2 \eta dx,$$

whence

$$\begin{aligned} \int_{\tau}^T \int_K |x|^{-2} |u_t|^2 dx dt &\leq \int_{\tau}^T \int_{\overline{K}} |x|^{-2} |u_t|^2 \eta dx dt \\ &= \frac{1}{2} \int_{\overline{K}} |\nabla u(x, \tau)|^2 \eta dx - \frac{1}{2} \int_{\overline{K}} |\nabla u(x, T)|^2 \eta dx \leq C, \end{aligned}$$

the last step coming from the first estimate (2.8). \square

In particular, we deduce from Lemma 2.5 that $|x|^{-1} u_t = |x| \Delta u \in L_{\text{loc}}^2(\mathbb{R}^N \times (0, \infty))$, which at its turn shows that $|x|^{-2} u_t, \Delta u \in L_{\text{loc}}^1(Q_*)$. Hence u is a strong solution.

Step 3. General data. For general data $u_0 \in L_2^1(\mathbb{R}^N)$, we use the contraction principle (2.3). Let $u_{0,n}$ be a sequence of functions in $C_0^\infty(\mathbb{R}^N \setminus \{0\})$ such that $u_{0,n} \rightarrow u_0$ in $L_2^1(\mathbb{R}^N)$ and u_n the corresponding solution with initial datum $u_{0,n}$. By (2.3), the sequence u_n is uniformly Cauchy in $C([0, \infty); L_2^1(\mathbb{R}^N))$, hence it converges to a limit $u \in C([0, \infty); L_2^1(\mathbb{R}^N))$ which is a weak solution (passing to the limit in the identity (2.2) is immediate). Moreover, we deduce from (2.6) that u is uniformly bounded on compact subsets of Q_* . On the other hand, we apply Lemma 2.5 for u_n and we obtain the estimates (2.8), (2.9) locally uniformly with respect to n in Q_* . It follows first that the limit u satisfies the bound (2.6), then (2.8), and finally, by redoing for u the last step in the proof of Lemma 2.5, we deduce that u also satisfies (2.9). Hence u is a strong solution. \square

3 Radially symmetric solutions. The transformations

We restrict ourselves first to radially symmetric solutions $u(r, t) = u(|x|, t)$, $r = |x|$, of Eq. (1.1). In this case, we introduce the following change of variable

$$u(|x|, t) = v(y, t), \quad y = \log |x| + (N - 2)t. \quad (3.1)$$

By a simple calculation, we notice that

$$u_t(r, t) = (N - 2)v_y(y, t) + v_t(y, t), \quad u_r(r, t) = \frac{1}{r}v_y(y, t), \quad u_{rr}(r, t) = \frac{1}{r^2}(v_{yy}(y, t) - v_y(y, t)),$$

hence, by replacing in (1.1), we obtain that v is a solution to the Cauchy problem for the one-dimensional heat equation

$$v_t(y, t) = v_{yy}(y, t), \quad v_0(y) = u_0(r). \quad (3.2)$$

for $(y, t) \in \mathbb{R} \times (0, \infty)$. We observe that $x = 0$ is mapped through (3.1) into $y \rightarrow -\infty$. We thus can use the previous transformation (3.1) in order to prove Theorems 1.1 and 1.2. But before doing that, let us also notice that, in dimensions $N \geq 3$ and also $N = 1$, there exists another transformation, appearing in a more general version in [16, Subsection 1.3.4],

$$u(r, t) = e^{z-t}w(z, t), \quad z = -\frac{N-2}{2}\log |x|, \quad (3.3)$$

again mapping radially symmetric solutions $u(r, t)$ of Eq. (1.1) into solutions $w(z, t)$ of (3.2), but with a different connection between initial data:

$$u_0(|x|) = e^z w_0(z) = |x|^{-(N-2)/2} w_0(z).$$

The change of variable and function (3.3) acts as an inversion for $N \geq 3$: it maps $x = 0$ to $y \rightarrow +\infty$ and $|x| \rightarrow +\infty$ to $y \rightarrow -\infty$, and it acts as a direct mapping only for $N = 1$. In the sequel, we will use mainly transformation (3.1) due to its simplicity, but at some points (3.3) will appear too.

4 Radially symmetric solutions. Asymptotic convergence

In this section we prove Theorems 1.1 and 1.2 for radially symmetric solutions and we deduce some more interesting remarks about the quite unexpected asymptotic behavior for Eq. (1.1).

Proof of Theorem 1.1. Let u be a solution for (1.1) with initial datum satisfying (1.5) and (1.6). By transformation (3.1), we arrive to a solution v of (3.2) with initial datum v_0 such that

$$\lim_{y \rightarrow \infty} v_0(y) = \lim_{y \rightarrow -\infty} v_0(y) = 0$$

and

$$\begin{aligned} M_{v_0} &:= \int_{-\infty}^{\infty} v_0(y) dy = \int_{-\infty}^{\infty} u_0(e^y) dy = \int_0^{\infty} \frac{u_0(r)}{r} dr \\ &= \int_0^{\infty} r^{-N} u_0(r) r^{N-1} dr = \frac{1}{\omega_1} \int_{\mathbb{R}^N} |x|^{-N} u_0(|x|) dx = \frac{M_{u_0}}{\omega_1} < \infty, \end{aligned} \quad (4.1)$$

where, as usual, ω_1 is the area of the unit sphere of \mathbb{R}^N . In fact, we notice that the same equality (4.1) holds true taken at any time $t > 0$ instead of $t = 0$. From the standard mass conservation property for the heat equation, we deduce that the quantity $\int_{\mathbb{R}^N} |x|^{-N} u(x, t) dx$ is conserved.

Due to well-known results in the theory of the heat equation, we find

$$\lim_{t \rightarrow \infty} t^{1/2} \left| v(y, t) - M_{v_0} \frac{1}{\sqrt{4\pi t}} e^{-y^2/4t} \right| = 0, \quad (4.2)$$

uniformly for $y \in \mathbb{R}$. We translate (4.2) in terms of u to get

$$\lim_{t \rightarrow \infty} t^{1/2} \left| u(|x|, t) - \frac{M_{u_0}}{\omega_1} \frac{1}{\sqrt{4\pi t}} \exp \left(-\frac{(\log |x| + (N-2)t)^2}{4t} \right) \right| = 0, \quad (4.3)$$

that is (1.7). \square

We prove now the asymptotic convergence for the case when $u_0(0) = K > 0$.

Proof of Theorem 1.3. (a) Let u be a solution for (1.1) with initial datum satisfying $u_0(0) = K > 0$, (1.5) and (1.13). By the transformation (3.1), we arrive to a solution v of (3.2) whose initial datum v_0 satisfies

$$\lim_{y \rightarrow \infty} v_0(y) = 0, \quad \lim_{y \rightarrow -\infty} v_0(y) = K > 0. \quad (4.4)$$

Moreover, we notice that

$$\begin{aligned} \int_{-\infty}^{\infty} |KH(y) - v_0(y)| dy &= \int_{-\infty}^0 |K - v_0(y)| dy + \int_0^{\infty} |v_0(y)| dy \\ &= \int_0^1 \frac{|K - u_0(r)|}{r} dr + \int_1^{\infty} \frac{u_0(r)}{r} dr \leq \frac{1}{\omega_1} I_1 < \infty, \end{aligned} \quad (4.5)$$

due to the conditions on u_0 in (1.13), where we have denoted by H the complementary Heaviside function

$$H(y) = \begin{cases} 1, & \text{for } y < 0, \\ 0, & \text{for } y \geq 0. \end{cases}$$

We now use the following

Lemma 4.1. *Let v_1, v_2 be two solutions of the one-dimensional heat equation with initial data satisfying*

$$\int_{-\infty}^{\infty} |v_2(y, 0) - v_1(y, 0)| dy < \infty. \quad (4.6)$$

Then we have

$$\|v_1(t) - v_2(t)\|_{\infty} = O(t^{-1/2}), \quad \text{as } t \rightarrow \infty. \quad (4.7)$$

The proof of Lemma 4.1 is immediate, since $w(y, t) := v_2(y, t) - v_1(y, t)$ is a solution for (3.2) with integrable initial datum, thus has the desired order as $t \rightarrow \infty$ by standard results for the heat equation.

We apply now Lemma 4.1 for the solutions $v(y, t)$ and $K \operatorname{erfc}(y, t)/2$ of (3.2). Their initial data are $v_0(y)$ and $KH(y)$ in the previous notations, thus they satisfy the condition (4.6), due to (4.5) and the fact that $I_1 < \infty$. Thus, we get that

$$\left\| v(t) - \frac{K}{2} \operatorname{erfc}(t) \right\|_{\infty} = O(t^{-1/2}), \quad \text{as } t \rightarrow \infty,$$

whence, undoing the transformation (3.1) and coming back to the initial variables, we deduce (1.11).

(b) Let u be a solution for (1.1) with initial datum satisfying $u_0(0) = K > 0$, (1.5), (1.13) and (1.14). By the transformation (3.1), we arrive to a solution v of (3.2) with initial datum v_0 satisfying (4.4). We make a further change by letting $w(y, t) := K - v(y, t)$, hence w is a solution to (3.2) with

$$\lim_{y \rightarrow \infty} w_0(y) = K > 0, \quad \lim_{y \rightarrow -\infty} w_0(y) = 0.$$

Let $\psi(y, t) := w_y(y, t) = -v_y(y, t)$. Then again ψ is a solution for (3.2), and we want to apply for ψ the following result:

Lemma 4.2. *Let ψ be a solution for Eq. (3.2), whose initial datum $\psi_0(y) := \psi(y, 0)$ satisfies the following conditions:*

$$M(\psi) := \int_{-\infty}^{\infty} \psi_0(y) dy \in (0, \infty), \quad \varrho(\psi) := \int_{-\infty}^{\infty} |y^3 \psi_0(y)| dy < \infty. \quad (4.8)$$

Then, the following two results hold true:

$$\|\psi(y, t) - M(\psi)G(y, t)\|_p = O(t^{-2+1/2p}), \quad \left\| \int_{-\infty}^y (\psi(s, t) - M(\psi)G(s, t)) ds \right\|_{\infty} = O(t^{-3/2}). \quad (4.9)$$

where $G(y, t)$ is the standard Gaussian

$$G(y, t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{y^2}{4t}\right).$$

Lemma 4.2 is proved in [14, Theorem 2]. There, it is asked that $\psi_0 \geq 0$, but it is easy to check that the result holds for any ψ bounded. We have to check that ψ satisfies the condition (4.8). We have:

$$M(\psi) = \int_{-\infty}^{\infty} \psi_0(y) dy = \lim_{y \rightarrow \infty} (w_0(y) - w_0(-y)) = K \in (0, \infty),$$

and

$$\begin{aligned} \varrho(\psi) &= \int_{-\infty}^{\infty} |y^3 v_y(y, 0)| dy = \int_0^{\infty} |(\log r)^3 r u_r(r, 0)| \frac{dr}{r} \\ &= \int_0^{\infty} |(\log r)^3 u_{0,r}(r)| dr = \frac{1}{\omega_1} \int_{\mathbb{R}^N} \frac{|(\log |x|)^3|}{|x|^{N-1}} |\nabla u_0(|x|)| dx < \infty, \end{aligned}$$

due to the condition (1.14). Thus, by Lemma 4.2, we obtain the asymptotic convergence for ψ as in (4.9), with $M(\psi) = K$. We keep the second result in (4.9) and transform it to get

$$\left\| w(y, t) - K \int_{-\infty}^y G(s, t) ds \right\|_{\infty} = O(t^{-3/2}),$$

or equivalently, recalling that $w(y, t) = K - v(y, t)$,

$$\left\| v(y, t) - K \int_y^{\infty} G(s, t) ds \right\|_{\infty} = O(t^{-3/2}). \quad (4.10)$$

Noticing that

$$\int_y^{\infty} G(s, t) ds = \frac{1}{\sqrt{4\pi t}} \int_y^{\infty} e^{-s^2/4t} ds = \frac{2\sqrt{t}}{2\sqrt{\pi t}} \int_{y/(2\sqrt{t})}^{\infty} e^{-z^2} dz = \frac{1}{2} \operatorname{erfc}\left(\frac{y}{2\sqrt{t}}\right),$$

and finally undoing transformation (3.1) to get back to the initial variables, we arrive to (1.15) as desired. \square

Remarks. (i) **Decay rates as $t \rightarrow \infty$.** For $x \neq 0$ fixed, the behavior as $t \rightarrow \infty$ of $E(x, t)$, $F(x, t)$ is different with respect to the dimension N . Indeed, when $N \geq 3$, both profiles decay to 0 as $t \rightarrow \infty$ with a rate

$$\frac{1}{\sqrt{t}} \exp\left(-\frac{(N-2)^2}{4}t\right),$$

as it is easy to check from the explicit formulas. Since the two special solutions make sense also for $N = 1$ and $N = 2$, we analyze the decay rate also in these cases. For $N = 2$, we have $\lim_{t \rightarrow \infty} E(x, t) = 1$

and $F(x, t) = O(1/\sqrt{t})$ as $t \rightarrow \infty$, and for $N = 1$ we have $\lim_{t \rightarrow \infty} E(x, t) = 2$ and $F(x, t)$ behaves in the same way as for $N \geq 3$.

(ii) **Self-map.** Consider Eq. (1.1) in dimensions N, \bar{N} respectively. Since both are mapped to solutions of (3.2) through their correspondent transformation (3.1), we have

$$y = \log |x| + (N - 2)t = \log |\bar{x}| + (\bar{N} - 2)t,$$

whence we obtain the mapping between radially symmetric solutions in the two dimensions

$$\bar{u}(\bar{x}, t) = u(x, t), \quad |x| = |\bar{x}|e^{(\bar{N}-N)t}. \quad (4.11)$$

The transformation (4.11) is a *self-map* of Eq. (1.1) between dimensions N and \bar{N} .

(iii) **Hotspots. Inner and outer regimes.** Since $F(0, t) = 0$ and $F(x, t) \rightarrow 0$ as $t \rightarrow \infty$, it is natural to think about the evolution of the maximum points of F at time t (called *hotspots*). At this point we use the second transformation (3.3), in order to get, in the new variables (z, t) ,

$$F(x, t) = \bar{F}(z, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(N-2)^2}{4}t} e^{z - \frac{z^2}{(N-2)^2 t}}.$$

We omit the intermediate calculations which are straightforward. Notice that the maximum lies at $z_0 = (N - 2)^2 t / 2$, and at that point, $F(x, t) = 1/\sqrt{4\pi t}$. Hence, the hotspots evolve with a decay rate which is smaller than the "linear" decay rate as $t \rightarrow \infty$. This is a typical feature of the presence of two different regimes of convergence, one for $|x|$ small (the so-called *inner regime*) and another one for $|x|$ large (the so-called *outer regime*). This phenomenon exists also in literature for the porous medium equation in domain with holes in dimension $N \geq 3$ [1] or for the p -Laplacian equation in dimension $N > p$ [9]. The difference between the behavior close to $x = 0$ and close to infinity will become more clear in the case $m > 1$ that will be studied in the forthcoming paper [8].

5 Asymptotic convergence for general solutions

We are now ready to prove Theorem 1.2 in its maximal generality.

Proof of Theorem 1.2. Let u be a solution of Eq. (1.1) with initial datum u_0 satisfying (1.5) and (1.10), with $u_0(0) = K > 0$. We construct the sub- and supersolutions $u_-(x, t)$, $u_+(x, t)$ that are the (radially symmetric) solutions of Eq. (1.1) having initial data

$$u_{-,0}(r) := \min\{u_0(x) : |x| = r\}, \quad u_{+,0}(r) := \max\{u_0(x) : |x| = r\}, \quad (5.1)$$

which are obviously continuous and bounded, and both satisfy (1.10), thus also (1.13). Thus, by the comparison principle, Corollary 2.4, we have

$$u_-(x, t) \leq u(x, t) \leq u_+(x, t), \quad (x, t) \in \mathbb{R}^N \times (0, \infty).$$

From part (a) of Theorem 1.3 we know that (1.11) holds true for u_- and u_+ . Since $u_{-,0}(0) = u_{+,0}(0) = K$, it follows that

$$\left| u(x, t) - \frac{K}{2} E(x, t) \right| \leq \max \left\{ \left| u_-(x, t) - \frac{K}{2} E(x, t) \right|, \left| u_+(x, t) - \frac{K}{2} E(x, t) \right| \right\} = O(t^{-1/2}),$$

whence u satisfies (1.11). \square

At this point we also prove Proposition 1.4.

Proof of Proposition 1.4. (a) Let u be a radially symmetric solution with $u_0 = M\delta_0$. Then, by (3.1), its transformed v will be a solution to (3.2), whose initial data will be the zero function, since $x = 0$ is mapped into $y \rightarrow -\infty$. But in our class of solutions, this solution of (3.2) is the zero function, by standard uniqueness results [6, Theorem 7, p. 58], which is a contradiction.

(b) Assume in a first step that u is radially symmetric. We map $u(r, t)$ into $v(y, t)$ by (3.1). Then $v_0 \equiv 0$ in $(-\infty, \log r_0)$. Due to standard heat equation theory [6, Section 2.3.3 (a)], $v(y, t) > 0$ for any $t > 0$, $y \in (-\infty, \log r_0)$, but it remains true that $\lim_{y \rightarrow -\infty} v(y, t) = 0$, for any $t > 0$. Coming back to u , we reach the conclusion.

Let now u be a non-radial strong solution. We define the radially symmetric functions u_- , u_+ as in (5.1). Since $u_-(r, t) \leq u(x, t)$, it follows that $u(x, t) > 0$ for any $x \neq 0$, $t > 0$. Since $u(x, t) \leq u_+(r, t)$, it follows that $u(0, t) = 0$, for any $t > 0$. \square

6 Extensions, comments and open problems

1. Dimension $N = 1$. In this case, the transformation (3.1) applies to any solution. We thus notice that the effect of the singularity at $x = 0$ is just disconnecting the real line. More precisely, let u be a continuous solution to (1.1) posed in dimension $N = 1$ and such that $u(0, t) = 0$ for all $t > 0$. Define

$$u_+(x, t) := \begin{cases} u(x, t), & \text{if } x > 0, \\ 0, & \text{if } x \leq 0, \end{cases} \quad u_-(x, t) := \begin{cases} u(x, t), & \text{if } x < 0, \\ 0, & \text{if } x \geq 0, \end{cases}$$

and notice that, applying the transformation (3.1) to both u_+ and u_- , we obtain two different solutions

$$v_+(y, t) := u_+(x, t), \quad y = \log x - t, \quad x > 0, \quad \text{and} \quad v_-(z, t) = u_-(x, t), \quad z = \log(-x) - t, \quad x < 0,$$

which will asymptotically converge towards two different Gaussian profiles with different masses M_+ and M_- as indicated in the proof of Theorem 1.1. Coming back to initial variables, the profile of u will be

$$U(x, t) = \begin{cases} M_+ F(x, t), & \text{if } x > 0, \\ M_- F(x, t), & \text{if } x < 0. \end{cases}$$

In particular, we give a simple explicit example of solution having two branches of the type indicated above:

$$F_1(x, t) := \begin{cases} \alpha F(x, t), & \text{for } x \leq 0, \\ (1 - \alpha) F(x, t), & \text{for } x > 0, \end{cases} \quad (6.1)$$

for any $\alpha \in (0, 1)$, $\alpha \neq 1/2$. We leave the details of the proofs of the previous statements to the reader, along the lines of the proof of Theorem 1.1. For initial data with $u_0(0) = K > 0$, Theorem 1.2 still applies in this case. In order to show the difference of behavior with respect to $N \geq 3$, we plot in Figure 2 the two typical profiles for dimension $N = 1$.

2. Improvements and open problems related to Theorems 1.2 and 1.3. Besides the conditions in Theorem 1.3, in the case $u_0(0) = K$, one can improve the rate of convergence to the asymptotic profile $E(x, t)$ by using, for example, results from the paper [21]. We leave these improvements to the reader, along the lines of the proof of Theorem 1.3. On the other hand, an open problem arises naturally from the results in Theorems 1.2 and 1.3. That is, can one prove (1.11) for **general** (that means, not necessarily radially symmetric) solutions, only asking the condition (1.13) to hold?

3. Asymptotic profiles of non-radially symmetric solutions with $u_0(0) = 0$. Theorem 1.1 and the counterexamples following it raise a very natural, but also in our opinion very difficult problem to classify the asymptotic profiles for general solutions of Eq. (1.1) when $u_0(0) = 0$. Since the counterexamples are constructed in a very specific way, a first partial question that could be raised is: are there all the profiles weighted combinations of the radial one? For dimension $N = 1$ we give an answer in the first comment of this section, but the problem remains open for the rest of dimensions.

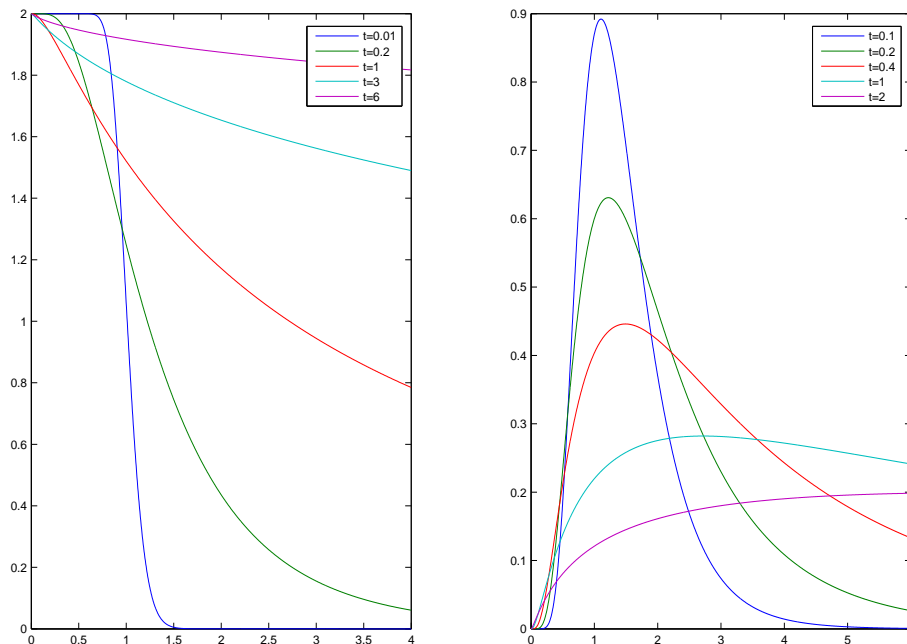


Figure 2: Profiles E and F for dimension $N = 1$ at various times.

4. General densities. As we have said in the Introduction, there exists an increasing interest for the more general problem (1.3). This case has not been yet studied in detail with $\varrho(x) \sim |x|^{-2}$ as $|x| \rightarrow \infty$ and $m = 1$, but it has been strongly studied for $m > 1$, [18, 19, 10] and references therein. The authors consider there densities ϱ which are regular near $x = 0$ and show that the asymptotic behavior is given by the fundamental solution to (1.4). It might be interesting to raise also the problem of considering general densities of various other forms, for example

$$\varrho(x) \sim |x|^{-2} \text{ as } |x| \rightarrow 0, \quad \lim_{|x| \rightarrow \infty} \varrho(x) = C \in (0, \infty),$$

that is, preserving the other property of the pure power density (the singularity at $x = 0$) and renouncing to the first one (the behavior as $|x| \rightarrow \infty$).

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